Helicity within the Kolmogorov phenomenology of turbulence

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In a phenomenology in which both energy and helicity exhibit net flux to the small scales it is natural to investigate how they might influence each other. Motivated by Kraichnan's 1971 derivation of spectral scaling laws using the timescale for energy transfer in wavenumber, we proceed by considering the timescale for helicity transfer and its potential impact on energy distribution in wavenumber. We demonstrate using resolved direct numerical simulations that the predicted effects of a second timescale related to helicity transfer are consistent with observed statistics. Both the energy and helicity spectra show to close to a $k^{-4/3}$ scaling in the higher wavenumber 'bottleneck' regime of the inertial range. The latter scaling is in agreement with our prediction for the scaling exponent based on a helicity-dependent timescale for twisting rather than shearing motions.

1 Introduction

There are two quadratic invariants of the inviscid Navier-Stokes equations for 3D incompressible flows; the total energy, and the total helicity, $H = \int \mathbf{u}(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x})d\mathbf{x}$, where $\mathbf{u}(\mathbf{x})$ is the velocity field at point $\mathbf{x}$, and $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ is the vorticity. Unlike energy, helicity is a sign-indefinite *pseudo*-scalar, and is parity breaking (or reflection-antisymmetric), that is, it changes sign if $\mathbf{x} \to -\mathbf{x}$ and $\mathbf{u} \to -\mathbf{u}$. A useful picture to keep in mind to visualize helicity is that of a fluid parcel moving along a helix; the 'handedness' of this configuration is immediately apparent. Fluid helicity appears in many phenomena at scales ranging from atmospheric to viscous dissipation scales. In fact it is difficult to conceive of a flow with identically zero helicity in all wavenumbers.

A statistical relation for velocity correlations related to helicity flux has been sought in various publications beginning in 1961 [1] and such a relationship now exists for antisymmetric velocity correlations which describe the helicity flux (see references [2]):

$$\langle (u_l(\mathbf{x}+\mathbf{r}) - u_l(\mathbf{x}))(\mathbf{u}_t(\mathbf{x}+\mathbf{r})\times\mathbf{u}_t(\mathbf{x})\rangle = \frac{2}{15}hr^2 \qquad (1)$$

where u_l is the component of velocity $\mathbf{u}$ along the separation vector $\mathbf{r}$, $\mathbf{u}_t = \mathbf{u} - u_l\hat{\mathbf{r}}$ is the transverse component of the velocity, h is the mean helicity dissipation rate. This law is derived under the assumption of local statistical isotropy and homogeneity. It is analogous to the Kolmogorov 1941 4/5-law for the third-order longitudinal structure functions [3]. While the latter was derived assuming reflection-symmetry of the flow, Eq. (1) requires reflection symmetry breaking due to the presence of helicity. The 2/15-law was verified for the first time in resolved numerical simulations in [4].

In a phenomenology in which both energy and helicity exhibit net flux to the small scales (forward cascade) it is natural to investigate how they might influence each other. Motivated by Kraichnan's [5] derivation of spectral scaling laws using the timescale for energy transfer in wavenumber, we proceed to consider a timescale for helicity transfer and its potential impact on energy distribution in wavenumber. It seems reasonable that the twisting character of helical motions will give them a different characteristic timescale than, say, energy transfer by shearing motions between two parcels of fluid. We follow the arguments in [6] to construct a phenomenology with *two* timescales, τ_E for pure energy transfer and the τ_H for pure helicity transfer rates. The latter timescale implies a $k^{-4/3}$ scaling distribution of both energy and helicity. In [4] we demonstrated using DNS that a second timescale might indeed be important in the dynamics. In that work, resolved Navier-Stokes data were acquired in a periodic box at resolutions of 512^3 with controlled helicity input, and 1024^3 with random helicity input. Both the energy and helicity spectra show to close to a $k^{-4/3}$ scaling in the higher end of the inertial range. The latter scaling is consistent with our prediction for the scaling exponent based on a helicity-dependent timescale for twisting rather than shearing motions. We postulate self-consistent arguments for how such a phenomenology might arrange to affect the energy spectrum as observed. The main results summarized in this paper are discussed in detail in references [4, 7].

2 The phenomenology of helicity and related statistics

2.1 A transfer timescale of helicity and its effect on spectral scaling

We recall from Ref. [5] the definition of the distortion time of an eddy of size $1/k$ where k is the wavenumber:

$$\tau_E^2 \sim \Big(\int_0^k E(p)p^2dp\Big)^{-1} \sim [E(k)k^3]^{-1} \qquad (2)$$

This estimate was shown to lead to the well-known scaling of the energy spectrum $E(k) \sim k^{-5/3}$. Ref. [6] showed that, assuming the helicity cascade was governed by the energy timescale τ_E, the helicity spectrum is expected to scale the same way, as $H(k) \sim k^{-4/3}$.

We can define an analogous timescale for distortions of an eddy due to helicity transfer [7]:

$$\tau_H^2 \sim \Big(\int_0^k |H(k)|k^2\Big)^{-1} \tag{3}$$

The distortion or shear corresponding to τ_H is different from the distortion corresponding to τ_E of Ref. [5]. If τ_H is allowed to govern the energy and helicity timescales then we can estimate that $E(k) \sim H(k) \sim k^{-4/3}$. The ratio of the two timescales,

$$\frac{\tau_E}{\tau_H} \sim \Big(\frac{|H(k)|}{2kE(k)}\Big)^{-1/2}.$$

In [7] we argue for the possibility that τ_H may be comparable (though slower) than τ_E thus allowing for $k^{-4/3}$ scaling of the spectra in the high wavenumbers. In the next section we present the numerical results which support our proposed phenomenology.

3 Simulations and results

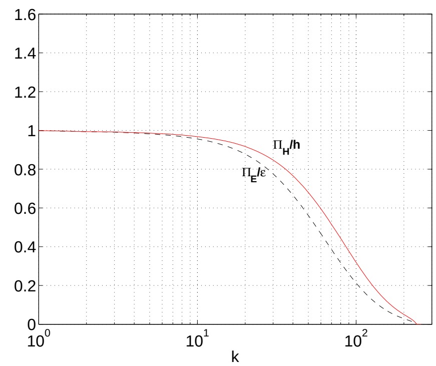
We will discuss two simulations in this section. The parameters of the simulations are listed in Table 1. Data I at resolution of 512^3 was forced with large (maximum) helicity input of the same sign at every timestep which resulted in a large mean helicity in statistically steady turbulence. Data II at resolution of 1024^3 was forced with random (uncontrolled) helicity at each timestep resulting in nearly zero mean helicity. Other details of the simulations, including the numerical schemes are given the primary references.

Table 1. Parameters of the numerical simulations I and II. ν - viscosity; R_λ - Taylor Reynolds number; mean total energy $E = \frac{1}{2}\sum_k |\tilde{\mathbf{u}}(\mathbf{k})|^2$; ε - mean energy dissipation rate; mean total helicity $H = \sum_k \tilde{\mathbf{u}}(\mathbf{k}) \cdot \tilde{\boldsymbol{\omega}}(-\mathbf{k})$; h - mean helicity dissipation rate; $\eta_\varepsilon = (\nu^3/\varepsilon)^{1/4}$; $\eta_h = (\nu^3/h)^{1/5}$.

	N	$\nu \times 10^4$	R_λ	E	ε	H	h	$\eta_\varepsilon \times 10^3$	$\eta_h \times 10^4$
I	512	1	270	1.72	1.51	26.8	62.2	9	1.7
II	1024	0.35	430	1.87	1.75	-0.12	13.2	4	1.3

In figure 1 we demonstrate using data I that both helicity and energy display constant fluxes (balanced by their respective dissipation rates) in about

Fig. 1. Fluxes of energy and helicity for data I. Dotted line: Flux of energy Π_E normalized by mean dissipation rate of energy ε. Solid line: Flux of helicity Π_H normalized by mean dissipation rate of helicity h.



the same range of scales. This is a demonstration of the simultaneous transfer of both energy and helicity to the small scales.

Figure 2 shows the spectra of energy and helicity for data I. In order to clearly distinguish between the slightly different scalings of $k^{-5/3}$ and $k^{-4/3}$, we plot on a linear-log scale and compensate the spectra by multiplying them by both $k^{5/3}$ (black lines) and $k^{-4/3}$ (blue dotted lines). Both energy and helicity spectra plotted in this way display no $k^{-5/3}$ scaling as predicted by the K41 theory. However, both spectra display agreement with $k^{-4/3}$ scaling in wavenumbers which are well-correlated with the constant flux regime in Fig. 1.

4 Conclusion

The $k^{-5/3}$ scaling law and the associated K41 phenemenology have underlying assumptions of statistically isotropic and reflection symmetric, helicity-free flows. Early efforts at integrating helicity into the picture, once it was discovered that it too is a global invariant, have usually relied on the (reasonable) guess that energy governs the overall dynamics with helicity being carried along more or less passively. Until now, the only accepted effect of helicity has been in slowing down the energy cascade in the following sense: if a flow is started with initially high helicity, the time it takes for the energy to reach the small scales is longer than if the flow were initialized with low or no helicity. The slowing down of energy transfer is thought to be benign in that the final statistical behavior of the flow, the spectral scalings and so forth, remain the same whether or not the flow is helical.

The phenomenology we propose here goes a bit further. First, we propose that it is not the net helicity but the relative (or local) helicity which is impor-

Fig. 2. Solid black lines – spectra compensated by $k^{5/3}$, dotted blue lines – same spectra compensated by $k^{4/3}$; horizontal lines indicate of the extent of scaling ranges. (a)Energy spectrum for the 512^3 simulation, which has a large mean helicity, there appears to be no scaling as $k^{-5/3}$, and very close to $k^{-4/3}$ scaling in the range $6 < k < 25$. (b)Helicity spectrum for the sam simulation.

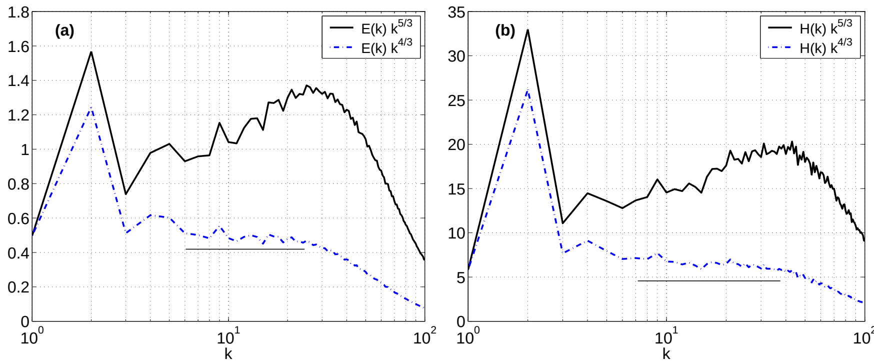
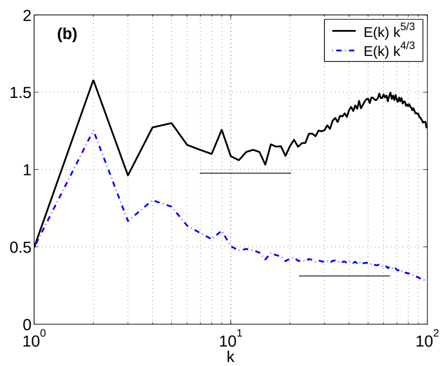


Fig. 3. Energy spectrum of data II represented in the same manner as figure 2a. Note the low-wavenumber scaling of $k^{-5/3}$ for a short range $8 < k < 20$ followed by a high wavenumber scaling of $k^{-4/3}$ for the range $20 < k < 70$. The two scaling ranges occur over comparable number of lengthscales and the latter scaling corresponds closely to the so-called 'bottleneck' feature.



tant in the dynamics. Second, the effect of relative helicity is locked into the dynamics in such a way as to cause energy to pile-up in the high-wavenumbers because of the slower timescale for helicity transfer. This results in the shallower $k^{-4/3}$ spectral scaling . The latter effect may also arise not from a cascade process, but because of generation of local helicity in the small scales by the local viscous production-dissipation of helicity. This issue can probably be verified with higher Reynolds numbers (lower viscosity) simulations to determine if this is a finite Reynolds number effect. The main conclusion of this work is that the bottleneck of energy in the high wavenumbers may be shown to be consistent with a phenomenology in which the presence of helicity slows down the energy cascade enough to cause a pile-up and hence a shallower spectral scaling.

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